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Probabilistic strength of glass cylinders subjected to flexure: total and local probabilities of fracture

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The parameters of the specific-risk-of-fracture function of Weibull were determined for glass cylinders subjected to flexure. The total probability of fracture was used considering the maximum stress of fracture, and considering also the location of fracture. Weibull's parameter m obtained by means of both procedures is in keeping with theory and it amounts to $m = 7.5$. In addition, the dispersion of Weibull's parameters was studied using the inversion of Fisher's information matrix, whose values showed a correct execution of testing. Moreover, the distribution of fracture time during testing was investigated, and neither fracture stresses nor location thereof exhibited whatever influence attributable to this time.

1. Introduction

The 3-point bending test of rectangular beams is exhibiting diverse sources of error in the determination of the maximum stress of fracture and hence in the determination of Weibull's parameters of the specific-risk-function [1]. One of these sources of error is constituted by the aleatory boundary conditions [2, 3]. For instance, the case of some rectangular beam requires parallelism among three straight lines, namely two lines of support and one line of loading, and in addition the contact between these lines and the material must be continuous. These boundary conditions modify the mean stresses of fracture when the properties of the material are deterministic, and if said properties are aleatory then fracture probability undergoes a modification. Reference [4, 5] study the change in Weibull's parameter m for terracotta bars subjected to traction and to flexotraction, making a fitting by means of Legendre's orthogonal polynomials, which did not answer parameter modification, and then considering that traction testing was excentrical and meant therefore a new condition of aleatory boundary. Now, in view of the foregoing, it is indeed very important to undertake testing in a manner that annuls aleatory boundary conditions. This is precisely what characterizes a bending test using round beams due to the absence of such conditions inasmuch as the beam has only three contact points with the testing device. The only thing that is to be corrected in this instance is the effect due to a punctual loading, that is to say the Seewald-Karman correction of deterministic nature. The effect of the Seewald-Karman correction has been treated already considering rectangular beam [6] and round beams [7] and, in the present case, this effect is of second order.

The evaluation of Weibull's parameters has been carried out already using sundry methods, for example least squares, maximum likelihood, and chi-squared [8], and such methods can be called analytical. Graphical methods have been also employed, and such nomographical procedures have been developed by León and Kittl [9] for the bending of rectangular beam, by Kittl *et al.* [10] for the bending of round beams, and by Díaz and Morales [11] for torsion. The estimation of parameters dispersion was made using Fisher's information matrix for diverse states of stress [6, 7, 12, 13]. The same method was employed by Trustrum [14] to estimate Weibull's parameter m considering local probability.

The present work aims at estimating Weibull's parameters in the bending of round beams of glass using the local and total probabilities of fracture, and estimating moreover the respective dispersions by means of Fisher's matrix, and estimating finally the distribution of fracture times.

2. Experimental procedure

90 test specimens of commercial glass 0.05 m long and 0.002 m radius were prepared and subjected to the 3-point bending test. The maximum stress of fracture σ along with fracture positions measured along the longitudinal axis of the beam as the least distance to one of the supporting points of the beam, were determined.

The commercial glass used here had the following composition: 74.32% SiO_2, 17.56% Na_2O, 1.38% K_2O, 0.11% MgO, 3.67% CaO, 2.35% Al_2O_3 and 0.14% Fe_2O_3.

The experimental results of both determinations supplied the maximum stress of fracture and the least distance between fracture and one of beam supporting points, which were ordered independently following an ascending order. This ordering allows to determine the probabilities considering the experimental data and using as estimator thereof the following relationship $(n-0.5)/N$ where n is the order number corresponding to stress or position values at most equal to a given value while N is the total number of tests.

The 3-point bending test was carried out applying the load gradually. The loading equipment was activated by means of a graduated system of weights, and the effect of time, coming from a some kind of static fatigue, was studied by recording the fracture time after the placing of the last loading weight. The weights were placed onto the loading equipment at intervals of 270 seconds. The fracture time was recorded for each of the 90 tests.

3. Total probability of fracture

According to the Probabilistic Strength of Materials [1, 2] the total cumulative probability of fracture of some isotropic and homogeneous solid subjected to a uniaxial and variable stress-field is as follows, considering surface brittleness:

$$F(\sigma) = 1 - \exp\left\{-\frac{1}{S_0}\int_S \phi[\sigma(r)]\,\mathrm{d}S\right\} \qquad (1)$$

where S_0 is surface unit, S is the surface of the material subjected to stress, r is position vector, $\sigma(r) \leq \sigma$ is stress-field, σ is maximum stress of fracture, and $\phi(\sigma)$ is Weibull's specific-risk-function. Equation 1 can be rewritten as follows:

$$\xi(\sigma) = \ln\frac{1}{1-F(\sigma)} = \frac{1}{S_0}\int_S \phi[\sigma(r)]\,\mathrm{d}S \qquad (2)$$

where $\xi(\sigma)$ is Evans function [15]. Function $\phi(\sigma)$ originally proposed by Weibull has the following analytical expression, potential and of two or three parameters:

$$\phi(\sigma) = \begin{cases} \left(\dfrac{\sigma - \sigma_L}{\sigma_0}\right)^m & \sigma_L < \sigma < \infty \\ 0 & 0 \leq \sigma \leq \sigma_L \end{cases} \qquad (3)$$

where m and σ_0 are parameters depending on the manufacturing process of the material whereas σ_L is the stress under which there is no fracture.

Now there will be considered a round beam, L long and of radius r, simply supported and subjected to a load P applied at the center of the bearing length. Then the stress-field in keeping with the elemental theory of beams, is:

$$0 \leq \sigma(x,y,z) = \frac{2xy}{Lr}\sigma \leq \sigma = \frac{PL}{\pi r^3}$$
$$0 \leq x \leq \frac{L}{2};\quad 0 \leq z = \sqrt{r^2 - y^2} \leq r;\quad 0 \leq y \leq r \qquad (4)$$

If Weibull's specific-risk-function includes two parameters, that is to say if $\sigma_L = 0$ in Equation 3, then the introduction of Equation 4 into Equation 2 yields:

$$\xi(\sigma) = \frac{Lr\sqrt{\pi}}{2S_0(m+1)}\left(\frac{\sigma}{\sigma_0}\right)^m \frac{\Gamma\left(\dfrac{m+1}{2}\right)}{\Gamma\left(\dfrac{m+2}{2}\right)} \qquad (5)$$

where Γ is the Euler gamma function:

$$\Gamma(m) = \int_0^\infty t^{m-1}\mathrm{e}^{-t}\,\mathrm{d}t \qquad (6)$$

4. Local probability of fracture

The probabilistic strength of materials allows to determine the probability with which some solid may fracture at a certain point thereof. The fundamental equation of the local probability of fracture is, according to [16]:

$$\frac{\mathrm{d}n(r)}{n} = \frac{\phi[\sigma(r)]\,\mathrm{d}S}{\int_s \phi[\sigma(r)]\,\mathrm{d}S} \qquad (7)$$

where $\mathrm{d}n(r)/n$ is the percentage of fracture initiated at the point r of surface $\mathrm{d}S$ and n is the total number of fractures.

For the case of some round beam subjected to flexure, the consideration of Equations 3 with $\sigma_L = 0$ and of Equation 4 leads to the transformation of Equation 7 into the following one:

$$\frac{n(x)}{n} = \int_0^x \frac{\mathrm{d}n(x)}{n} = \frac{1}{\int_s \phi[\sigma(r)]\,\mathrm{d}S} \times \int_0^x \phi[\sigma(r)]\,\mathrm{d}S = \left(\frac{2x}{L}\right)^{m+1} \qquad (8)$$

where x is the least distance measured between the point of fracture and one of the supporting points.

Note that, in general, for a something material subjected to flexure it is necessary to determine, moreover, the location, below neutral axis, where the inside crack growing until fracture. However, in this particular case, that it is not necessary because in glasses subjected to flexure the cracks that give origin to fracture grows in the free surface of the material.

5. Dispersion of the parameters

Fisher's information matrix R is determined using the following equation, according to [12]:

$$r_{ij} = -nE\left(\frac{\partial^2 \ln f(\sigma;\theta)}{\partial\theta_i \partial\theta_j}\right)$$
$$\{\theta\} = \{m, \sigma_0\} \tag{9}$$

where E is the expected-value operator, f is probability-density function, and n is sample size. If Weibull's specific-risk function includes two parameters then function $f(\sigma)$ can be written, in the case of flexure and total probability of fracture, as follows:

$$f(\sigma) = K\frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1}\exp\left\{-K\left(\frac{\sigma}{\sigma_0}\right)^m\right\} \tag{10}$$

where K is defined as follows in keeping with Equation 5:

$$K = \frac{Lr\sqrt{\pi}}{2S_0(m+1)}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \tag{11}$$

Hence the elements r_{ij} of Fisher's matrix are:

$$\begin{aligned}
r_{11} &= -nE\left(\frac{\partial^2 \ln f(\sigma;m,\sigma_0)}{\partial m^2}\right) = n\left(\frac{1}{K}\frac{\partial K}{\partial m}\right)^2 \\
&\quad + \frac{n}{m^2}(1.82379 - 0.84555\ln K + \ln^2 K) \\
r_{12} &= -nE\left(\frac{\partial^2 \ln f(\sigma;m,\sigma_0)}{\partial m \partial\sigma_0}\right) \\
&= -\frac{n}{\sigma_0}\left(\frac{m}{K}\frac{\partial K}{\partial m} - \ln K + 0.42277\right) \\
r_{22} &= -nE\left(\frac{\partial^2 \ln f(\sigma;m,\sigma_0)}{\partial\sigma_0^2}\right) = n\left(\frac{m}{\sigma_0}\right)^2
\end{aligned} \tag{12}$$

where $\frac{\partial K}{\partial m}$ is defined by:

$$\begin{aligned}
\frac{\partial K}{\partial m} &= \frac{Lr\sqrt{\pi}}{2S_0(m+1)}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \\
&\quad \times\left\{\left[\frac{\Gamma'\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma'\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right] - \frac{1}{m+1}\right\}
\end{aligned} \tag{13}$$

in which Γ' is the derivate of the Euler gamma function.

The matrix of variances and covariances is easily determined by means of the inversion of Fisher's matrix, with the due consideration that r_{11} be positive. Hence the dispersion of parameter θ_i is $v_{ii}^{1/2}$ if $V = R^{-1}$. The expressions of the variances and covariances are determined considering:

$$\begin{aligned}
\mathrm{Var}(m) &= \frac{r_{22}}{r_{11}r_{22} - r_{12}^2} \\
\mathrm{Var}(\sigma_0) &= \frac{r_{11}}{r_{11}r_{22} - r_{12}^2} \\
\mathrm{Co-Var}(m,\sigma_0) &= \frac{r_{12}}{r_{12}^2 - r_{11}r_{22}}
\end{aligned} \tag{14}$$

For local probability of fracture there is only one parameter, m, in consecuence the Fisher's matrix have one element. In accord with Equation 8 the probability density-function for this case is:

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{n(x)}{n}\right] = \frac{2(m+1)}{L}\left(\frac{2x}{L}\right)^m \tag{15}$$

Hence, considering Equation 9 the element of Fisher's matrix is:

$$r_{11} = -nE\left(\frac{\partial^2 \ln f(x;m)}{\partial m^2}\right) = \frac{n}{(m+1)^2} \tag{16}$$

and, the variance of parameter m for local probability of fracture is

$$\mathrm{Var}(m) = \frac{(m+1)^2}{n} \tag{17}$$

6. Analysis of the results

The experimental data corresponding to the maximum stresses of fracture permitted to construct the respective Weibull diagram of $\ln\xi(\sigma)$ versus $\ln\sigma$. The consideration of Equation 5 allows to write:

$$\ln\xi(\sigma) = m\ln\sigma + \ln\frac{Lr\sqrt{\pi}}{2S_0(m+1)}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \tag{18}$$

Now, in a Weibull diagram, shown in Fig. 1, the above Equation 18 is represented by a straight line whose slope supplies the value of parameter m.

The diagram of the total cumulative probability of fracture corresponding to $n(x)/n$ versus $2x/L$ was constructed using the experimental datas regarding the least distance between the point of fracture and one of the supportint points in the 3-point test of bending. Equation 8 supplies the following relationship:

$$\ln\frac{n(x)}{n} = (m+1)\ln\frac{2x}{L} \tag{19}$$

In a diagram, shown in Fig. 2, the above Equation 19 is represented by a straight line whose slope amounts to $(m+1)$ and supplies thus the value of parameter m.

Inasmuch as testing has affected 90 commercial glass beams of like manufacture, so that the fabrication procedure has been maintained constant for all these beams,

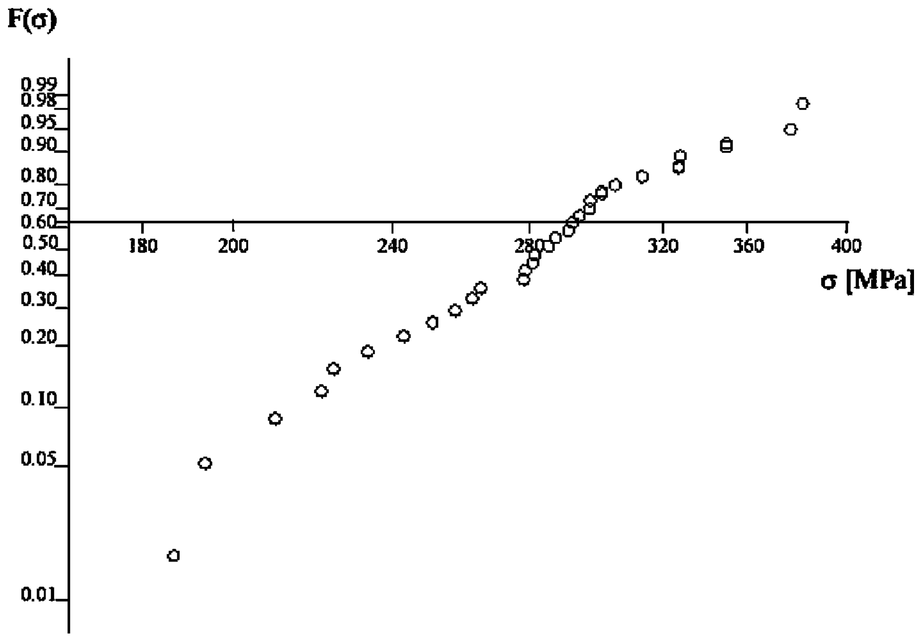

Figure 1 Weibull diagram of total cumulative probability of fracture.

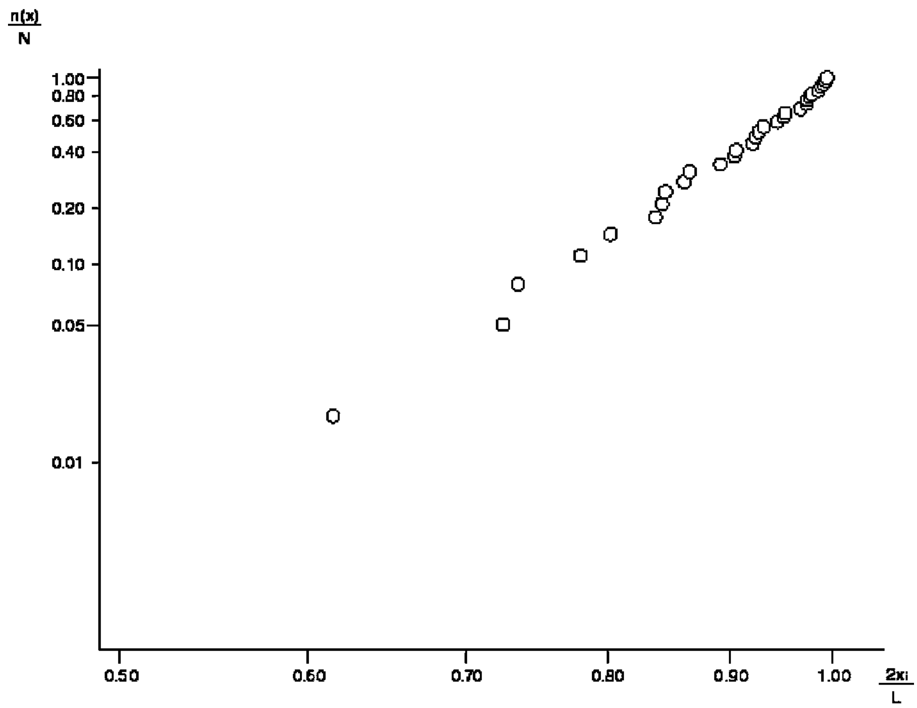

Figure 2 Diagram of local probability of fracture.

the corresponding parameter of Weibull must be the same, independently of its determination through the total probability or through the local probability using in both instances the 3-point test of bending.

Table I given below shows the results of the 3-point test of bending. This table supplies Weibull parameters m and σ_0 for the total probability of fracture, which were determined in a first approximation through a least-square regression and paremeter m for the local probability of fracture which was estimated too through a least-square regression. The dispersions were determined by means of Fisher's matrix Equation 14 and Equation 17 for total probability of fracture and local probability of fracture, respectively, and were included in Table I.

TABLE I Weibull parameters and their dispersion in the bending of 90 round glass beams

	Weibull parameters	
Probability	m	σ_0 (MPa)
Total	7.5 ± 0.6	21 ± 4
Local	7.5 ± 0.9	

It can be readily noticed in above Table I that the values of parameter m determined for the local probability and for the total probability are practically equal. This result is showing the advantages of employing round test specimens in the 3-point test of bending because such a procedure affords and independence from the problem constituted by the aleatory boundary conditions and allows thus to completely avoid such a problem.

If the least distance between the point of fracture and a supporting point is graphically plotted as a function of time in a fashion not shown herein, then a complete aleatoric condition, i.e. randomness, can be observed. Therefore fracture-time influence does not play whatever role in the statistical variables studied herein and concerning the strength and the position of fractures.

7. Conclusions

It has been possible to get an excellent result in the comparison of the values of Weibull parameter m, achieving a complete agreement between the theory and the experience gathered, through the determination of value $m = 7.5$ in keeping with the total and local probabilities of fracture. The foregoing is the fruit of the use of round beams for undertaking the 3-point bending test in a manner that is affording a complete control over the aleatory boundary conditions. The small value of dispersion, both in parameter m and in parameter σ_0, for total probability of fracture—namely 0.6 and 4, respectively—and 0.9 for local probability of fracture, is showing that the tests were properly carried out. As concerns fracture times, they exhibited a complete randomness, and there was observed no influence at all of these times on fracture stresses and on the minimun distances between fracture points and one of the supporting points.

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